7.5 Apply Properties of Logarithms

Before
You evaluated logarithms.

Now
You will rewrite logarithmic expressions.

Why?
So you can model the loudness of sounds, as in Ex. 63.

Key Vocabulary
• base, p. 10

Key Concept
Properties of Logarithms
Let \( b, m, \) and \( n \) be positive numbers such that \( b \neq 1 \).

Product Property
\[ \log_b mn = \log_b m + \log_b n \]

Quotient Property
\[ \log_b \frac{m}{n} = \log_b m - \log_b n \]

Power Property
\[ \log_b m^n = n \log_b m \]

Example 1
Use properties of logarithms
Use \( \log_4 3 \approx 0.792 \) and \( \log_4 7 \approx 1.404 \) to evaluate the logarithm.

a. \[ \log_4 \frac{3}{7} = \log_4 3 - \log_4 7 \]
\[ \approx 0.792 - 1.404 \]
\[ = -0.612 \]
Quotient property
Use the given values of \( \log_4 3 \) and \( \log_4 7 \).
Simplify.

b. \[ \log_4 21 = \log_4 (3 \cdot 7) \]
\[ = \log_4 3 + \log_4 7 \]
\[ \approx 0.792 + 1.404 \]
\[ = 2.196 \]
Product property
Use the given values of \( \log_4 3 \) and \( \log_4 7 \).
Simplify.

Power property

Avoid Errors
Note that in general
\[ \log_b \frac{m}{n} \neq \log_b m - \log_b n \]
and
\[ \log_b mn \neq (\log_b m)(\log_b n). \]

Guided Practice
for Example 1
Use \( \log_6 5 \approx 0.898 \) and \( \log_6 8 \approx 1.161 \) to evaluate the logarithm.

1. \( \log_6 \frac{5}{8} \)
2. \( \log_6 40 \)
3. \( \log_6 64 \)
4. \( \log_6 125 \)
REWRITING EXPRESSIONS You can use the properties of logarithms to expand and condense logarithmic expressions.

**Example 2** Expand a logarithmic expression

Expand \( \log_6 \frac{5x^3}{y} \).

\[
\log_6 \frac{5x^3}{y} = \log_6 5x^3 - \log_6 y \\
= \log_6 5 + \log_6 x^3 - \log_6 y \\
= \log_6 5 + 3 \log_6 x - \log_6 y
\]

**Example 3** TAKS PRACTICE: Multiple Choice

Which of the following is equivalent to \( \log 3 + 3 \log 4 \) – \( \log 6 \)?

- A) \( \log 6 \)
- B) \( \log 8 \)
- C) \( \log 32 \)
- D) \( \log 61 \)

**Solution**

\[
\log 3 + 3 \log 4 \) – \( \log 6 = \log 3 + \log 4^3 - \log 6 \\
= \log (3 \cdot 4^3) - \log 6 \\
= \log \frac{3 \cdot 4^3}{6} \\
= \log 32
\]

- The correct answer is C. A) B) C) D)

**Guided Practice** for Examples 2 and 3

5. Expand \( \log 3x^4 \).

6. Condense \( \ln 4 + 3 \ln 3 - \ln 12 \).

**Change-of-Base Formula** Logarithms with any base other than 10 or \( e \) can be written in terms of common or natural logarithms using the change-of-base formula. This allows you to evaluate any logarithm using a calculator.

**Key Concept**

**Change-of-Base Formula**

If \( a, b, \) and \( c \) are positive numbers with \( b \neq 1 \) and \( c \neq 1 \), then:

\[
\log_c a = \frac{\log_b a}{\log_b c}
\]

In particular, \( \log_c a = \frac{\log a}{\log c} \) and \( \log_c a = \frac{\ln a}{\ln c} \).
EXAMPLE 4 Use the change-of-base formula

Evaluate \( \log_3 8 \) using common logarithms and natural logarithms.

Solution

Using common logarithms:

\[
\log_3 8 = \frac{\log 8}{\log 3} \approx \frac{0.9031}{0.4771} = 1.893
\]

Using natural logarithms:

\[
\log_3 8 = \frac{\ln 8}{\ln 3} \approx \frac{2.0794}{1.0986} = 1.893
\]

EXAMPLE 5 Use properties of logarithms in real life

SOUND INTENSITY For a sound with intensity \( I \) (in watts per square meter), the loudness \( L(I) \) of the sound (in decibels) is given by the function

\[
L(I) = 10 \log \frac{I}{I_0}
\]

where \( I_0 \) is the intensity of a barely audible sound (about \( 10^{-12} \) watts per square meter). An artist in a recording studio turns up the volume of a track so that the sound’s intensity doubles. By how many decibels does the loudness increase?

Solution

Let \( I \) be the original intensity, so that \( 2I \) is the doubled intensity.

Increase in loudness = \( L(2I) - L(I) \)

\[
= 10 \log \frac{2I}{I_0} - 10 \log \frac{I}{I_0}
\]

\[
= 10 \left( \log 2 + \log \frac{I}{I_0} - \log \frac{I}{I_0} \right)
\]

\[
= 10 \log 2
\]

\[
= 3.01
\]

\( \text{The loudness increases by about 3 decibels.} \)

GUIDED PRACTICE for Examples 4 and 5

Use the change-of-base formula to evaluate the logarithm.

7. \( \log_5 8 \) \hspace{1cm} 8. \( \log_9 14 \) \hspace{1cm} 9. \( \log_{26} 9 \) \hspace{1cm} 10. \( \log_{12} 30 \)

11. WHAT IF? In Example 5, suppose the artist turns up the volume so that the sound’s intensity triples. By how many decibels does the loudness increase?
1. **VOCABULARY**  Copy and complete: To condense the expression \( \log_3 2x + \log_3 y \), you need to use the _____ property of logarithms.

2. **WRITING** Describe two ways to evaluate \( \log_7 12 \) using a calculator.

**MATCHING EXPRESSIONS** Match the expression with the logarithm that has the same value.

3. \( \ln 6 - \ln 2 \)  
4. \( 2 \ln 6 \)  
5. \( 6 \ln 2 \)  
6. \( \ln 6 + \ln 2 \)

A. \( \ln 64 \)  
B. \( \ln 3 \)  
C. \( \ln 12 \)  
D. \( \ln 36 \)

**APPROXIMATING EXPRESSIONS** Use \( \log 4 \approx 0.602 \) and \( \log 12 \approx 1.079 \) to evaluate the logarithm.

7. \( \log 3 \)  
8. \( \log 48 \)  
9. \( \log 16 \)  
10. \( \log 64 \)

**EXPANDING EXPRESSIONS** Expand the expression.

11. \( \log 144 \)  
12. \( \log \frac{1}{3} \)  
13. \( \log \frac{1}{4} \)  
14. \( \log \frac{1}{12} \)

**ERROR ANALYSIS** Describe and correct the error in expanding the logarithmic expression.

31. \( \log_2 5x = (\log_2 5)(\log_2 x) \)

32. \( \ln 8x^3 = 3 \ln 8 + \ln x \)

**CONDENSING EXPRESSIONS** Condense the expression.

33. \( \log_4 7 - \log_4 10 \)  
34. \( \ln 12 - \ln 4 \)

35. \( 2 \log x + \log 11 \)  
36. \( 6 \ln x + 4 \ln y \)

37. \( 5 \log x - 4 \log y \)  
38. \( 5 \log_4 2 + 7 \log_4 x + 4 \log_4 y \)

39. \( \ln 40 + 2 \ln \frac{1}{2} + \ln x \)  
40. \( \log_5 4 + \frac{1}{3} \log_5 x \)

41. \( 6 \ln 2 - 4 \ln y \)  
42. \( 2(\log_3 20 - \log_3 4) + 0.5 \log_3 4 \)

43. **MULTIPLE CHOICE** Which of the following is equivalent to \( 3 \log_4 6 \)?

A. \( \log_4 18 \)  
B. \( \log_4 72 \)  
C. \( \log_4 216 \)  
D. \( \log_4 256 \)
44. **MULTIPLE CHOICE** Which of the following statements is **not** correct?

(A) \( \log_3 48 = \log_3 16 + \log_3 3 \)  
(B) \( \log_3 48 = 3 \log_3 2 + \log_3 6 \)  
(C) \( \log_3 48 = 2 \log_3 4 + \log_3 3 \)  
(D) \( \log_3 48 = \log_3 8 + 2 \log_3 3 \)

45. Use the change-of-base formula to evaluate the logarithm.

(a) \( \log_4 7 \)  
(b) \( \log_5 13 \)  
(c) \( \log_3 15 \)  
(d) \( \log_8 22 \)

49. \( \log_3 6 \)  
50. \( \log_5 14 \)  
51. \( \log_6 27 \)  
52. \( \log_8 32 \)

53. \( \log_6 \frac{24}{5} \)  
54. \( \log_2 \frac{15}{7} \)  
55. \( \log_3 \frac{9}{40} \)  
56. \( \log_7 \frac{3}{16} \)

61. **ERROR ANALYSIS** Describe and correct the error in using the change-of-base formula.

\[ \log_3 7 = \frac{\log 3}{\log 7} \]

**SOUND INTENSITY** In Exercises 62 and 63, use the function in Example 5.

62. Find the decibel level of the sound made by each object shown below.

(a) Barking dog: \( I = 10^{-4} \text{ W/m}^2 \)  
(b) Ambulance siren: \( I = 10^0 \text{ W/m}^2 \)  
(c) Bee: \( I = 10^{-6.5} \text{ W/m}^2 \)

63. The intensity of the sound of a trumpet is \( 10^3 \) watts per square meter. Find the decibel level of a trumpet.

64. **OPEN-ENDED MATH** For each statement, find positive numbers \( M, N, \) and \( b \) (with \( b \neq 1 \)) that show the statement is false in general.

(a) \( \log_b (M + N) = \log_b M + \log_b N \)  
(b) \( \log_b (M - N) = \log_b M - \log_b N \)

**CHALLENGE** In Exercises 65–68, use the given hint and properties of exponents to prove the property of logarithms.

65. **Product property** \( \log_b mn = \log_b m + \log_b n \)  
(Hint: Let \( x = \log_b m \) and let \( y = \log_b n \). Then \( m = b^x \) and \( n = b^y \).)

66. **Quotient property** \( \log_b \frac{m}{n} = \log_b m - \log_b n \)  
(Hint: Let \( x = \log_b m \) and let \( y = \log_b n \). Then \( m = b^x \) and \( n = b^y \).)

67. **Power property** \( \log_b m^n = n \log_b m \)  
(Hint: Let \( x = \log_b m \). Then \( m = b^x \) and \( m^n = b^{nx} \).)

68. **Change-of-base formula** \( \log_b a = \frac{\log_a a}{\log_c c} \)  
(Hint: Let \( x = \log_b a, y = \log_b c, \) and \( z = \log_c a \). Then \( a = b^x, c = b^y, \) and \( a = c^z \), so that \( b^x = c^z \).)
69. CONVERSATION Three groups of people are having separate conversations in a room. The sound of each conversation has an intensity of \(1.4 \times 10^{-3}\) watts per square meter. What is the decibel level of the combined conversations in the room?

70. PARKING GARAGE The sound made by each of five cars in a parking garage has an intensity of \(3.2 \times 10^{-4}\) watts per square meter. What is the decibel level of the sound made by all five cars in the parking garage?

71. ★ SHORT RESPONSE The intensity of the sound TV ads make is ten times as great as the intensity for an average TV show. How many decibels louder is a TV ad? Justify your answer using properties of logarithms.

72. BIOLOGY The loudest animal on Earth is the blue whale. It can produce a sound with an intensity of \(10^{6.8}\) watts per square meter. The loudest sound a human can make has an intensity of \(10^{0.8}\) watts per square meter. Compare the decibel levels of the sounds made by a blue whale and a human.

73. ★ EXTENDED RESPONSE The f-stops on a 35 millimeter camera control the amount of light that enters the camera. Let \(s\) be a measure of the amount of light that strikes the film and let \(f\) be the f-stop. Then \(s\) and \(f\) are related by the equation:

\[s = \log_2 f^2\]

a. Use Properties Expand the expression for \(s\).

b. Calculate The table shows the first eight f-stops on a 35 millimeter camera. Copy and complete the table. Describe the pattern you observe.

<table>
<thead>
<tr>
<th>(f)</th>
<th>1.414</th>
<th>2.000</th>
<th>2.828</th>
<th>4.000</th>
<th>5.657</th>
<th>8.000</th>
<th>11.314</th>
<th>16.000</th>
</tr>
</thead>
</table>

c. Reasoning Many 35 millimeter cameras have nine f-stops. What do you think the ninth f-stop is? Explain your reasoning.
74. **CHALLENGE** Under certain conditions, the wind speed \(s\) (in knots) at an altitude of \(h\) meters above a grassy plain can be modeled by this function:

\[ s(h) = 2 \ln(100h) \]

a. By what factor does the wind speed increase when the altitude doubles?

b. Show that the given function can be written in terms of common logarithms as 
\[ s(h) = \frac{2}{\log e} (\log h + 2). \]

---

**Mixed Review for TAKS**

**TAKS Practice** Which of the following is *not* an example of a Pythagorean triple? **TAKS Obj. 10**

- A 8, 15, 17
- B 48, 64, 80
- C 7, 23, 25
- D 11, 60, 61

**TAKS Practice** Which inequality best describes the range of the function whose graph is shown? **TAKS Obj. 2**

- F \(y \leq -1\)
- G \(y \leq 3\)
- H \(y \geq -1\)
- J \(y \geq 3\)

---

**Quiz for Lessons 7.4–7.5**

Evaluate the logarithm without using a calculator. **(p. 499)**

1. \(\log_4 16\)
2. \(\log_5 1\)
3. \(\log_8 8\)
4. \(\log_{1/2} 32\)

Graph the function. State the domain and range. **(p. 499)**

5. \(y = \log_2 x\)
6. \(y = \ln x + 2\)
7. \(y = \log_3 (x + 4) - 1\)

Expand the expression. **(p. 507)**

8. \(\log_2 5x\)
9. \(\log_5 x^7\)
10. \(\ln 5xy^3\)
11. \(\log_3 \frac{6y^4}{x^8}\)

Condense the expression. **(p. 507)**

12. \(\log_3 5 - \log_3 20\)
13. \(\ln 6 + \ln 4x\)
14. \(\log_6 5 + 3 \log_6 2\)
15. \(4\ln x - 5\ln x\)

Use the change-of-base formula to evaluate the logarithm. **(p. 507)**

16. \(\log_3 10\)
17. \(\log_7 14\)
18. \(\log_5 24\)
19. \(\log_8 40\)

20. **SOUND INTENSITY** The sound of an alarm clock has an intensity of \(I = 10^{-4}\) watts per square meter. Use the model \(L(I) = 10 \log \frac{I}{I_0}\), where \(I_0 = 10^{-12}\) watts per square meter, to find the alarm clock’s loudness \(L(I)\). **(p. 507)**

---

**Extra Practice** for Lesson 7.5, p. 1016

**Online Quiz**

**Apply Properties of Logarithms**
7.5 **Graph Logarithmic Functions**

**TEKS** a.5, a.6, 2A.11.B

**QUESTION** How can you graph logarithmic functions on a graphing calculator?

You can use a graphing calculator to graph logarithmic functions simply by using the \( \log \) or \( \ln \) key. To graph a logarithmic function having a base other than 10 or \( e \), you need to use the change-of-base formula to rewrite the function in terms of common or natural logarithms.

**EXAMPLE** Graph logarithmic functions

Use a graphing calculator to graph \( y = \log_2 x \) and \( y = \log_2 (x - 3) + 1 \).

**STEP 1** **Rewrite functions** Use the change-of-base formula to rewrite each function in terms of common logarithms.

\[
\begin{align*}
y &= \log_2 x \\
&= \frac{\log x}{\log 2} \\
y &= \log_2 (x - 3) + 1 \\
&= \frac{\log (x - 3)}{\log 2} + 1
\end{align*}
\]

**STEP 2** **Enter functions** Enter each function into a graphing calculator.

\[
\begin{align*}
Y1 &= \frac{\log X}{\log 2} \\
Y2 &= \frac{\log (X-3)}{\log 2} + 1
\end{align*}
\]

**STEP 3** **Graph functions** Graph the functions.

**PRACTICE** Use a graphing calculator to graph the function.

1. \( y = \log_4 x \)  
2. \( y = \log_8 x \)  
3. \( f(x) = \log_3 x \)  
4. \( y = \log_5 x \)  
5. \( y = \log_{12} x \)  
6. \( g(x) = \log_9 x \)  
7. \( y = \log_3 (x + 2) \)  
8. \( y = \log_5 x - 1 \)  
9. \( f(x) = \log_4 (x - 5) - 2 \)  
10. \( y = \log_2 (x + 4) - 7 \)  
11. \( y = \log_7 (x - 5) + 3 \)  
12. \( g(x) = \log_3 (x + 6) - 6 \)  
13. **REASONING** Graph \( y = \ln x \). If your calculator did not have a natural logarithm key, explain how you could graph \( y = \ln x \) using the \( \text{LOG} \) key.